On Spectral Analysis of Directed Graphs with Graph Perturbation Theory

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Abstract—**The eigenspace of the adjacency matrix of a graph possesses important information about the network structure. Recent works showed the existence of line orthogonality patterns when nodes are projected into the eigenspaces of graphs by either SVD or regular eigen decomposition. Further study proved why such phenomena hold true for simple graphs using matrix perturbation theory and several graph partition algorithms were introduced. However, the majority of the studies only focused on undirected graphs. In this paper, we will explore the adjacency eigenspace properties for directed graphs. With the aid of the graph perturbation theory, we emphasize on deriving rigorous mathematical results to explain several phenomena related to the eigenspace projection patterns for directed graphs. Furthermore, we will relax our original assumption and generalize the theories to the perturbed Perron-Frobenius simple invariant subspace so that the theories can adept to a much broader range of network structural types. Our theories have the potential for various applications such as graph partition for directed graphs.**

I. Introduction

Spectral graph analysis has received significant attentions in research community. There is a large literature on examining the eigenvectors of the adjacency matrix or its variants (Laplacian or normal matrix) for networks with various applications such as spectral clustering [1]–[6] , spectral pattern analysis [7], [8] and graph visualization [9]. However, most of previous works only based their studies on simple undirected graphs.

For many graphs generated from online social networks, economic networks or biological networks, we usually get the information with directions. For example, follow and retweet data in Twitter, import or export data for global trade, and food web data generated from food chain in the ecosphere can all be expressed by a directed graph rather than a simple graph. Such directed graphs contain rich information than the simplified undirected graphs.

Researchers have developed various methods by transforming the adjacency matrix into a symmetric one, or creating some relationship or affinity matrix containing edge information. The most common method is to remove the edge directions and treat the graph as undirected. A notable work [10] extended the modularity optimization based community detection method into directed graphs. Another approach to deal with the directed graph is to transform the affinity matrix that is constructed using the edge direction information into a symmetric matrix, then community detection methods based on undirected graphs can be used to do the clustering [11], [12]. However, those transformation approaches would either lose the important information of the directions of data or introduce some noise to the data which would influence our analysis in a negative way.

When the concern is with directed graphs, one of the main difficulties for spectral clustering is to deal with the complex values for both eigenvalues and eigenvectors. The mixture of real and complex eigenvalues or the complete absence of real eigenvalues made the analysis of the eigenspace properties for such graphs a very complicated problem. Since real and complex numbers are defined in two separate planes, the direct comparison of any entities from those two sets could not produce a meaningful result in terms of the nodes' geometric positions in spectral space. Hence, in this paper, we will provide a theoretical analysis of the properties of the eigenspace for directed graphs and propose a method to circumvent the issue of complex eigenpairs. We will utilize the connectedness property of the components of a network to screen our irrelevant eigenpairs and thus eliminating the need for dealing with complex eigenpairs. Then, it is possible to select appropriate eigenparis for spectral projections.

Our theoretical analysis is based on the well studied graph perturbation theory which focuses on analyzing the changes in the spectral space of a graph after new edges are added or deleted. The work [6] demonstrated how the perturbation theory can be applied to explain phenomena such as orthorgonal lines and tilting rays in undirected graphs. However, the case is more complicated for directed graphs, since the simple linear combinations of eigenvectors method used for symmetric matrices can no longer produce reliable approximations for the perturbed eigenvectors of asymmetric matrices. In section III, we will discuss the details about those issues and propose a certain type of orthonormal basis based on which the main theoretical results derived in section IV can produce a relatively simple math representation. Therefore, we aim to provide a theoretical analysis of the properties of the eigenspace for directed graphs using graph perturbation theory and propose a method to circumvent the issues of complex eigenpairs and non-simple eigenvalues.

In the next step, the derived theories were generalized to perturbed Perron-Frobenius simple invariant subspace.

The significance of such a spectral subspace is that it is a real subspace with some unique properties that contains all the spectral clustering information of a graph. If selected correctly, all the nodes can be partitioned to their corresponding communities correctly. This generalization can be used to explain the behavior of the nodes in the spectral projection space and to aid developing more efficient spectral clustering algorithms. Although some previous works based on undirected graphs also utilized the properties of such a perturbed simple invariant subspace, none of these works generalized the results to state this point. Because all the eigenvectors are real and can form an orthonormal basis automatically in symmetric adjutancy matrices, the perturbed eigenspaces can be approximated directly with the linear combinations of eigenvectors. Therefore, simple spectral structures of undirected graphs concealed the existence and significance of such a simple invariant subspace.

However, finding such a simple invariant subspace using algorithmic approaches is not easy due to the complex situations affecting the values and signs of the vectors forming this subspace. Details for such analysis will be discussed in section IV and V. Nevertheless, our theoretical study revealed some useful properties unique to such a subspace which we can leverage to aid the search for such a subspace.

A. *Our Contributions*

The obvious problem is that spectral clustering in directed graphs need to deal with the complex valued eigenvectors and eigenvalues. Theoretical analysis is also difficult, since the eigenvectors do not form orthonormal basis automatically like those in undirected graphs. Furthermore, since the differences among the eigenvalues of directed graphs are not Lipschitz continuous, the eigenvalues of asymmetric matrices are not as well conditioned as those in undirected graphs. Therefore, most researchers tried to circumvent those problems either by treating the graph as undirected by removing the edge directions or by constructing some relationship/affinity matrices that transform the original adjacency matrices to symmetric forms. The first type of methods discard important edge information. The second type of methods are computationally intensive for large graphs.

We will start our theoretical work by investigating the structures of communities and those of networks in section II. Section III includes a proposed orthonormal basis which can be used to derive the approximations for the eigenvectors of our interest. Then, we will use the structural information of graphs to generalize several properties that existed in the spectral spaces of directed networks in section IV A and section IV B. In section IV C, we will use the graph perturbation theory to justify such properties. Section IV D discusses perturbation influences to eigenvectors. The first difficulty for theoretical analysis could be avoided by introducing the Core of a community definition in section IV E, and the second one can be overcame by applying matrix perturbation theory to our constructed orthonormal basis from section III in section V. The final section concludes this work and some future topics are proposed.

The main contributions of this work are:

- Derived theories and mathematical approximations to analyze the projections' behaviors of nodes in the spectral space behave for directed graphs
- Relieved the assumptions of sizes of communities and the existence of eigenvalue gaps for components
- Introduced the concept of the Core of a community so that the theories and the partitioning algorithm can be extended to more general situations
- Generalized the theories to introduce the perturbed Perron-Frobenius invariant subspace, which contains the key graph partitioning information

II. Preliminaries

In this paper we only consider binary graphs without self-loops. A directed graph can be represented as the adjacency matrix $A_{n\times n}$ with $a_{ij} = 1$ if there exists an edge pointing from node V_i to node V_j and $a_{ij} = 0$ if such directed edge was not present. A directed(undirected) graph is called strongly connected(connected) if there exists a path for any nodes V_i to V_j of the graph $G(V,E)$. All the theories and results derived in this work also apply to undirected graphs.

Symbols and Definitions	
A	Adjacency matrix
P	Permutation matrix
$G(V,E)$	Graph of A
$\widetilde{A}$	Perturbed matrix of A
$\mathcal{L}(L)$	The set of eigenvalues of L
$\Re(X)$	An invariant subspace of A spanned by X, which is part of a basis of A
$\rho(A)$	The spectral radius of A
$(q_1,\cdots,q_n)$	An Orthonormal basis of A
$(\lambda_1,\cdots,\lambda_n)$	Eigenvalues of A
A^H	Conjugate transpose of A
Unitary	A is unitary if $A^{-1} = A^H$

There are many ways to define what is a community in a network. We impose the assumption that communities in a directed graph at a global level should be those that satisfy the following properties: the graph of a community is strongly connected and the size of a community which is proportional to the corresponding spectral radius should be relatively large in size. However, from the section IV E, we will replace this assumption with a much weaker one to extend the applicability of our theories and make the graph partitioning method derived from the theories more practically usable.

A. *Reducibility and Connectedness of a Graph*

A graph is called reducible if the adjacency matrix A of the graph can be conjugated into block upper triangular block form by a permutation matrix P. A graph is called irreducible if this statement does not hold. When there exists several irreducible subgraphs within a graph as stated in [13], then we call the following equation as the normal form of a reducible matrix:

$$PAP^{-1} = \begin{pmatrix} A_1 & & E \\ & \ddots & \\ \mathbf{0} & & A_K \end{pmatrix}, \tag{1}$$

where A_is are either irreducible or zeros and E contains the edges connecting A_is. In the case where all the entries in E are 0s, then A is conjugated into the diagonal block form.

The following lemma stated in [13] establishes an important relationship between the connectedness and reducibility of a graph:

Lemma 1. *Let A be the adjacency matrix of a directed(undirected) graph $G(V, E)$, then A is strongly connected(connected) iff G is irreducible.*

B. Implications of Perron-Frobenius Theorem

Based on our community assumptions, it becomes possible for us to identify the eigenpairs that are closely tied to the properties of corresponding communities and reveal important topological information of graphs. We will be able to achieve such a task with the relevant parts of the Perron-Frobenius theorem for non-negative irreducible matrices in chapter 8 of [14]:

Lemma 2. *Let A be an irreducible and non-negative $n \times n$ matrix. Let $\lambda_1, \cdots, \lambda_n$ be the (real or complex) eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. Then its spectral radius $\rho(A)$ is defined as:*

$$\rho(A) \stackrel{\text{def}}{=} \max_i(|\lambda_i|) \tag{2}$$

This spectral radius is called the Perron-Frobenius eigenvalue and it has the following properties:

1) *The number $\rho(A)$ is a positive real number and it is a simple eigenvalue of A.*
2) *The only eigenvector that has all positive components is the one associated with $\rho(A)$. All the eigenvectors associated with other eigenvalues will have mixed signed components.*

The direct result we get from Lemma 2 is that it provides us a guideline to select the eigenvectors corresponding to the spectral radii of all the communities to form a subspace for spectral projection while screening out the incomparable complex eigenvectors. The reason that the Perron-Frobenius eigenvectors are real is because that, for any real eigenvalue, we can solve for a real eigenvector, although complex solutions may also exist.

C. Eigenspace Projection

Before we can perform the eigenspace projection, we need to make sure that the eigenvectors forming the spectral space are linearly independent, or else the eigenspace will not be of full rank. With this assumption, we will adept the eigenspace projection method as in the work [6]. The difference is that they worked on symmetric adjacency matrices with the spectral spaces formed by orthonormal basis. In our work, we only require the eigenvectors to be linearly independent. An illustration of the projection method is given as (3). The eigenvector $\boldsymbol{x}_i$ is represented as a column vector. The eigenvectors $\boldsymbol{x}_i$ ($i = 1, \cdots, K$) corresponding to the largest K Perron-Frobenius eigenvalues contain most topological information of the corresponding K largest communities of the graph in the spectral space. The K-dimensional spectral space is spanned by $(\boldsymbol{x}_1, \cdots \boldsymbol{x}_K)$. When a node u is projected in the K-dimensional subspace with $\boldsymbol{x}_i$ as the basis, the row vector $\boldsymbol{\alpha}_u = (x_{1u}, x_{2u}, \cdots, x_{Ku})$ are its coordinates in this spectral subspace.

$$\begin{array}{c} \\ \\ \\ \boldsymbol{\alpha}_u \rightarrow \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_i & \boldsymbol{x}_K & \boldsymbol{x}_n \\ & \downarrow & & \end{array} \\ \left(\begin{array}{cc|c|ccc} x_{11} & \cdots & x_{i1} & \cdots\ x_{K1} & \cdots & x_{n1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \hline x_{1u} & \cdots & x_{iu} & \cdots\ x_{Ku} & \cdots & x_{nu} \\ \hline \vdots & & \vdots & \vdots & & \vdots \\ x_{1n} & \cdots & x_{in} & \cdots\ x_{Kn} & \cdots & x_{nn} \end{array}\right) \end{array} \tag{3}$$

D. Perturbation Theory for Square Matrices

Spectral perturbation analysis studies the change of the eigenpairs when the graph is perturbed. It is an excellent mathematical tool for analyzing the influence of changes to the graph spectra. Hence, we will use it to estimate the spectral projections after perturbations and verify various phenomena in the adjacency eigenspace.

For a square matrix A with a perturbation E, the matrix after perturbation can be written as $\widetilde{A} = A + E$. Let λ_i be an eigenvalue of A with its eigenvector $\boldsymbol{x}_i$. In this paper, we assume that all the eigenvectors are normalized. For the perturbed matrix, we let $\tilde{\lambda}_i$ and $\tilde{\boldsymbol{x}}_i$ denote the perturbed eigenpairs. Under the assumption that the eigenvectors of A are linearly independent, it has been shown in chapter V of [15] that $\tilde{\boldsymbol{x}}_i$ can be estimated by the linear combination of $\boldsymbol{x}_i$ and the rest of the basis of A forming the invariant subspace $\Re(\boldsymbol{x}_i)^{\perp}$ of A. We reference the relevant definitions and theorems from the book as follows:

Lemma 3. *Let the columns of X be linearly independent and let columns of Y span $\Re(X)^{\perp}$. Then $\Re(X)$ is an invariant subspace of A if and only if $Y^H AX = 0$. In this case $\Re(Y)$ is an invariant subspace of A^H.*

Lemma 4. *Let $\Re(X)$ be an invariant subspace of A, columns of X form an orthonormal basis for $\Re(X)$, and (X, Y) be unitary. Then the decomposition of A has the reduced form:*

$$(X, Y)^H A(X, Y) = \begin{pmatrix} L_1 & H \\ \mathbf{0} & L_2 \end{pmatrix}, \tag{4}$$

where $L_1 = X^H AX$, $L_2 = Y^H AY$, and $H = X^H AY$. Furthermore, $AX = XL_1$, and the eigenvalues of L_1

are the eigenvalues of A associated with $\Re(X)$. The rest eigenvalues of A are those of L_2.

Definition 1. *Let $\Re(X)$ be an invariant subspace of A, and let (4) be its reduced form with respect to the unitary matrix (X, Y). Denote $\mathcal{L}(L)$ as the set of the eigenvalues of L. Then $\Re(X)$ is a simple invariant subspace if $\mathcal{L}(L_1) \cap \mathcal{L}(L_2) = \emptyset$.*

With the above definition and lemmas, under the assumption of simple invariant subspace, the approximations of perturbed eigenvectors are given by theorem 2.7 in chapter V of [15] as follows:

Lemma 5. *For a perturbed matrix $\widetilde{A} = A + E$, let $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ be a basis of A and denote $X = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K)$ and $Y = (\boldsymbol{x}_{K+1}, \cdots, \boldsymbol{x}_n)$. Suppose that (X, Y) is unitary, and suppose that $\Re(X)$ is a simple invariant subspace of A so that it has the reduced form as (4). For $i \in (1, \cdots, K)$, each perturbed eigenvectors $\tilde{\boldsymbol{x}}_i$ can be approximated as:*

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + Y(\lambda_i I - L_2)^{-1} Y^H E \boldsymbol{x}_i, \tag{5}$$

when the following conditions hold:

1) $\delta = \inf_{\|P\|=1} \|PHP\|_2 - \|X^H E X\|_2 - \|Y^H E Y\|_2 > 0$, *where* $H = X^H A Y$ *and* $p_i \approx (\lambda_i I - L_2)^{-1} Y^H E \boldsymbol{x}_i$ *for each* i.
2) $\gamma = \|Y^H E X\|_2 < \frac{1}{2}\delta$.

The H matrix in the above theorem can be uniquely solved when X is a simple invariant subspace of A so that $(\lambda_i I - L_2)^{-1}$ will have a solution. The details for solving this type of linear system were known as the Sylvester's equation problem and were introduced in the works [16] and [17]. In the next section, we will propose an orthonormal basis with an explicit form that can be used to derive approximations for perturbed eigenvectors.

III. Estimation of Perturbed Eigenvectors Using Orthonormal Basis

We mentioned that there exist some difficulties prohibiting the the direct applications of graph perturbation theory to asymmetric matrices. If graph perturbation theory is to be used to analyze the spectral spaces of directed graphs, the most difficult problem is that the perturbed eigenvectors cannot be estimated directly by other eigenvectors using simple linear combinations, since the eigenvectors do not form orthonormal basis automatically like those in undirected graphs. This difficulty has been solved by working with spectral resolutions in the work [18]. Another problem is that the differences among the eigenvalues of directed graphs are not Lipschitz continuous. In the work [17], the author used an orthogonal reduction to block triangular form to overcome the first problem and introduced function $\|\cdot\|_{sep}$ to overcome the second problem. Therefore, the estimations for perturbed eigenvectors can be expressed by the spectral resolution of A with respect to its simple invariant subspaces. These works lead to the orthonormal reduction method referenced in the previous section.

Therefore, with the help of Lemma 5, we know that under certain conditions we can still make good estimations of perturbed eigenvectors using the orthonormal reduction technique. To clarify the definitions, a basis that is orthogonal and of unit length for each vector spanning it is called an orthonormal basis. In the case when eigenvectors are orthogonalized and scaled to unit length, those eigenvectors form an orthonormal basis for the matrix. In a symmetric matrix, such a basis can diagonalize the matrix. However, for an asymmetric matrix, same process will result in an upper triangular matrix according to Schur's Theorem. Another notable fact is that although due to how eigenvectors are orthogonalized, the scalar -1 could be multiplied to the vectors and cause the signs of their components to flip, but this does not alter the properties of those eigenvectors, since the entire basis can be rotated and there is no concept of directions in the spectral space.

Hence, in order to make use of the above results to give explicit approximations used in the later sections to derive the theoretical results explaining the spectral projection patterns observe from directed graphs, we need to find an unitary orthonormal basis that satisfies the conditions mentioned above to achieve the orthonormal reduction for a given asymmetric matrix. Note that there exist an infinite number of sets of orthonormal basis that can achieve the same purpose. In general, any basis that satisfies the conditions in Lemma 3 and Definition 1 that can reduce the matrix into the block triangular reduced form in Equation 4 will work. However, the one we chose here helps us to derive relatively simpler mathematical representations for the perturbed eigenvectors in the next section. The following proposition sets up such an orthonormal basis by using the Gram-Schmidt process.

Proposition 1. *Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ be the set of eigenvalues for $\tilde{A}$, $\mathbf{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ be the corresponding set of eigenvectors, assume that all the eigenvectors are linearly independent, and, without loss of generality, let $Q = (\boldsymbol{q}_1, \cdots, \boldsymbol{q}_{i-1}, \boldsymbol{q}_{i+1}, \cdots, \boldsymbol{q}_n)$ be an orthonormal basis formed by Gram-Schmidt process. Suppose that some eigenvectors $\boldsymbol{q}_i = \boldsymbol{x}_i$ for $i \in (1, \cdots, K)$ are part of this orthonormal basis and denote $X = (\boldsymbol{q}_1, \cdots, \boldsymbol{q}_K)$. If λ_is are simple, then the following results hold:*

1) *$(X, Q)^H = (X, Q)^{-1}$ is unitary. $Q^H A X = 0$, thus $\Re(X)$ is a simple invariant subspace of A.*
2) *A can be reduced to a block triangular form:*

$$(X, Q)^H A (X, Q) = \begin{pmatrix} L_1 & H \\ \mathbf{0} & L_2 \end{pmatrix}, \tag{6}$$

where $L_1 = X^H A X$, $L_2 = Q^H A Q$ is upper triangular, and $H = X^H A Q$. $AX = XL_1$, The eigenvalues of L_1 are the eigenvalues of A associated with $\Re(X)$. The rest eigenvalues of A are those of L_2.

As stated in item 3 of the above Proposition 1, different from that of a symmetric matrix, the orthornormal reduction will result in an upper triangular matrix for an asymmetric matrix based on our selection of basis. In the symmetric case, the result is a diagonal matrix containing only eigenvectors, since the eigenvectors form an orthonormal basis directly and they will diagonalize the matrix.

Therefore, for the asymmetric case, if some eigenvectors $\boldsymbol{x}_i$ are part of the the orthonormal basis of A, then by Proposition 1, we can derive the approximations for the perturbed eigenvectors as follows:

Theorem 1. *For $i \in (1, \cdots, K)$, let the eigenvectors $\boldsymbol{x}_i$ be part of the the orthonormal basis of A and Q be the rest of the orthornaomal basis as constructed in Proposition 1. Assume that the conditions in Lemma 5 hold. Then, the perturbed eigenvectors $\tilde{\boldsymbol{x}}_i$ can be approximated as:*

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + \nabla E \frac{\boldsymbol{x}_i}{\lambda_i}, \tag{7}$$

where $\nabla = Q(I - \frac{L_2}{\lambda_i})^{-1} Q^H$.

In the next section, we will explore the spectral projection space and give the approximations for the perturbed Perron-Frobenius eigenvectors corresponding to the K communities that span the K dimensional spectral subspace. The approximations will be used to analyze several phenomena in this particular subspace.

IV. Eigenspace Properties of Directed Graphs

We start our theoretical work with the assumption that the graphs of communities are strongly connected, then we will replace it with a weaker one in the later part of this section. The observed graph A of a network with K communities namely $C_1, \cdots, C_K$, A can be considered as the perturbation of the matrix with disconnected communities by links connecting each other as E. Hence, the observed graph in the form of (1), can be regarded as the perturbation form the diagonal block form as:

$$A = \begin{pmatrix} A_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_K \end{pmatrix}, \tag{8}$$

where each A_i is the adjacency matrix of each community and E containing all the edges connecting communities. The permutation matrix P has the same effect of relabeling the nodes, so for simplicity, we will consider all the nodes relabeled as needed in this paper. Therefore, the observed adjacency matrix A has the form of (1).

In order to explore the adjacency eigenspace properties, we could look into the sub-eigenspace spanned by the eigenvectors of A corresponding to the Perron-Frobenius eigenvalues. The reason why we choose it is that, according to Lemma 2, the Perron-Frobenius eigenpairs are the spectral radii and corresponding eigenvectors of communities with various unique properties that no other eigenpairs possess. Thus, we can leverage such properties to derive a spectral clustering method for partitioning graphs.

A. Networks with Disconnected Communities

Given a network with K disconnected communities $C_1, \cdots, C_K$, the adjacency matrix of the network can be expressed in the form of (8), where each A_i is the adjacency matrix of community C_i. Then, we have the following results about the adjacency matrix A of the network:

Lemma 6. *For an adjacency matrix A of a graph with K disconnected communities in the form of (8). For $i = 1, \cdots, K$, the following results hold:*

1) *The K Perron-Frobenius eigenvalues λ_{Ci}s corresponding to communities C_is are real, positive, simple eigenvalues, and are also the eigenvalues of A.*
2) *Furthermore, let $\boldsymbol{x}_{C_i}$ be the Perron-Frobenius eigenvectors of communities, the eigenvectors $\mathbf{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K)$ of A corresponding to λ_{Ci}s are the only eigenvectors whose non-zero components are all positive, all the entries of $\mathbf{x}$ are real valued and have the following form:*

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_K) = \begin{pmatrix} \boldsymbol{x}_{C_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{x}_{C_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{x}_{C_K} \end{pmatrix}.$$

Proof: Since the matrix A is of diagonal block form, the eigenvectors of A will be of the the same form corresponding to each block and the eigenvalues of A will be the union of those of A_is. The results follow from applying Lemma 1 and Lemma 2. ∎

The direct result from Lemma 6 is that there is only one location of the row vector $\boldsymbol{\alpha_u}$ that has a non-zero value with the form:

$$\boldsymbol{\alpha_u} = (0, \cdots, 0, \boldsymbol{x}_{iu}, 0, \cdots, 0). \tag{9}$$

The location of x_{iu} indicates the community which node u belongs to and the value of x_{iu} denotes the influence of node u to that community. Since the graph is directed, the more out/in edge ratio a node has, the higher the value of x_{iu} will be. This observation coincides with the eigenvector centrality measure method. We will demonstrate the above findings in the following toy example. If we perform the eigenspace projection as 3, nodes from different communities will form different lines that are orthogonal to each other.

B. The Eigenspace of A Perturbed Graph

To illustrate various phenomena in the perturbed Perron-Frobenius spectral subspace, we generated a toy graph with 25 nodes containing 3 communities namely $C1$, $C2$ and $C3$. $C1$ contains nodes labeled 1 to 8, 14 and 15. $C2$ contains nodes 10, 12 and 13 with nodes 9 and 11 as leaf nodes. $C3$ is a stand alone community that contains the rest of nodes labeled 16 to 25. After adding directed edges between node 10 and 5 as perturbations, we created the graph in Figure 1.

When we look into its Perron-Frobenius spectral subspace, before the perturbation, the nodes form straight lines according to the communities they belong to and the coordinates of nodes from different communities form orthogonal lines. This phenomenon coincides with that of an undirected graph as mentioned in [6] which leads to our assumption that the pure line-orthogonality patterns in adjacency eigenspace projection was caused by the diagonal block structure of the adjacency matrix.

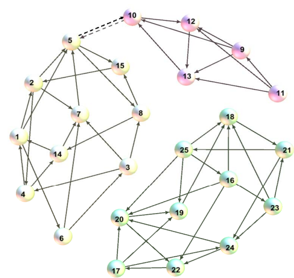
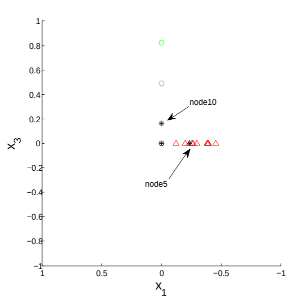
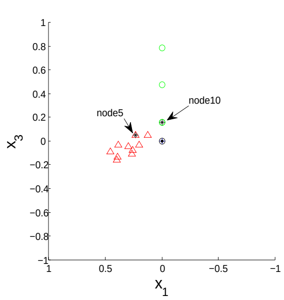


Fig. 1: Perturbed Graph

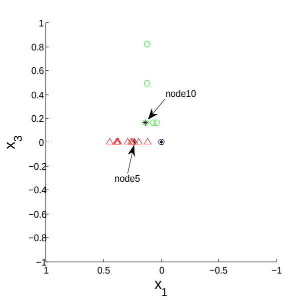
In order to thoroughly examine the spectral properties of nodes, we first added an edge from node 10 to node 5, then replaced it with a reverse edge, and finally added an undirected edge. The changes to the original isolated components are treated as as perturbations. We observed that, after the perturbations, the spectral coordinates of nodes changed in a consistent manner.



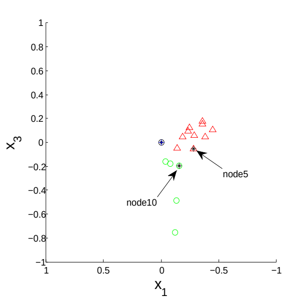
(a) Before Perturbation 2-D



(b) After Perturbation 10 to 5



(c) After Perturbation 5 to 10



(d) After Perturbation 5 and 10

Fig. 2: Spectral Coordinates of Nodes Before and After Perturbation 2-D

Figure 2 demonstrates the cross sectional spectral projections for communities $C1$ and $C2$ before and after perturbations. It can be observed clearly that the spectral coordinates of the nodes in the two communities that were connected by the edge have changed. However, different from those in undirected graphs as demonstrated in [6], the nodes in the community with an out-going edge are leaning towards the community that the edge is pointing to. On the other hand, the spectral coordinates of nodes in the community with an incoming edge still lie on their original axis but the value for each node on that axis has changed.

These observations demonstrated that the spectral projections behave more dynamically for directed graphs, since the information contained in edge directions are included. Furthermore, those observations confirmed our assumption that nodes' activities such as follow a Tweet, vote for a person or malicious attacks on a community can be clearly revealed in the spectral space for directed graphs. On the other hand, such behaviors are often concealed and blended in with some others in an undirected graph. Hence, analyzing the adjacency eigenspace of a directed graph reveals more about the its structure. Another notable phenomenon is that, when an undirected edge is added, the plot appeared to resemble the combination of both plots of adding one directed edge. The only difference in the plots is that the signs of some eigenvectors are flipped. The reasons behind such phenomenon is rather complicated, so a detailed analysis will be presented in section IV D, after all the key theoretical results are finished. Some of the subtle changes of spectral coordinates mentioned above may be difficult to observe from the figures directly, so we also provide the spectral coordinates corresponding to the related nodes from before and after the perturbations in table I for a better view.

TABLE I: Eigenvectors before and after perturbation

	no edge			edge 5 to 10		edge 10 to 5		edges 5 and 10	
node	x_1	x_2	x_3	$\tilde{x_1}$	$\tilde{x_2}$	$\tilde{x_1}$	$\tilde{x_2}$	$\tilde{x_1}$	$\tilde{x_2}$
1	-0.4570	0.0000	0.0000	0.4443	0.0000	-0.4570	0.0914	-0.4464	-0.1095
2	-0.3846	0.0000	0.0000	0.3738	0.0000	-0.3846	0.0332	-0.3833	-0.0491
3	-0.3957	0.0000	0.0000	0.3847	0.0000	-0.0.3957	0.0579	-0.3585	-0.1808
4	-0.3897	0.0000	0.0000	0.3788	0.0000	-0.3897	0.1330	-0.3595	-0.1520
5	-0.2346	0.0000	0.0000	0.2281	0.0000	-0.2346	-0.0499	-0.2835	0.0513
6	-0.2666	0.0000	0.0000	0.2592	0.0000	-0.2666	0.1081	-0.2439	-0.1222
7	-0.1210	0.0000	0.0000	0.1177	0.0000	-0.1210	-0.0499	-0.1390	0.0468
8	-0.2563	0.0000	0.0000	0.2491	0.0000	-0.2563	0.0748	-0.2328	-0.0930
9	0.0000	0.4932	0.0000	0.1254	0.4932	0.0000	-0.4738	-0.1336	0.4857
10	0.0000	0.1644	0.0000	0.1364	0.1644	0.0000	-0.1579	-0.1575	0.1941
11	0.0000	0.8220	0.0000	0.1198	0.8220	0.0000	-0.7896	-0.1219	0.7518
12	0.0000	0.1644	0.0000	0.0363	0.1644	0.0000	-0.1579	-0.0379	0.1615
13	0.0000	0.1644	0.0000	0.0704	0.1644	0.0000	-0.1579	-0.0772	0.1770
14	-0.2983	0.0000	0.0000	0.2899	0.0000	-0.2983	0.0416	-0.2870	-0.0572
15	-0.1984	0.0000	0.0000	0.1929	0.0000	-0.1984	-0.0332	-0.1879	-0.0448
16-25	0.0000	0.0000	0.xxxx	0.0000	0.0000	0.0000	-0.0000	0.0000	0.0000

As shown in the table and figures, we noticed that the signs of the components in the eigenvector corresponding to community $C2$ have all changed from non-negative to non-positive. Additionally, another phenomenon observed from the above results is that although nodes 9 and 11 were not members of any strongly connected component, but they could still be clustered into the strongly connected component which they pointed directly to with edges. We will analyze and explain the causes of these phenomena observed above with theories and utilize these phenomena to further relax some of our assumptions for partitioning a graph in the following sections.

C. Observed Graphs as Perturbations

In reality, the observed graphs with K loosely connected communities will take the form of (1). If we treat the observed graph as the perturbed graph from Equation 8, we can use the properties of Lemma 2, Lemma 6, Proposition 1 and Theorem 1 to derive the approximation of the perturbed Perron-Frobenius subspace as follows:

Theorem 2. *Let the observed graph be $\widetilde{A} = A + E$ with K communities and the perturbation E denotes the edges amongst communities $C_1, \cdots, C_K$. Assume that E satisfies the conditions in Lemma 5. The spectral subspace corresponding to the Perron-Frobenius eigenvalues is a simple invariant subspace of A, the perturbed eigenvectors for the observed graph $\tilde{A}$ can be approximated as:*

$$(\tilde{\boldsymbol{x}}_1, \cdots, \tilde{\boldsymbol{x}}_K) \approx (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K) + \nabla E(\frac{\boldsymbol{x}_1}{\lambda_1}, \cdots, \frac{\boldsymbol{x}_K}{\lambda_K}). \tag{10}$$

When spectral projection is performed based on the sub-eigenspace spanned by the eigenvectors of A corresponding to the K Perron-Frobenius eigenvalues, we can use Theorem 2 to derive the approximation of spectral coordinates of $\boldsymbol{\alpha}_u$ using the following simplified result that only takes into account of the influences of neighboring nodes from other communities. Since directed graphs are different from undirected graphs given that the edge direction indicates the flow of information, we define the outer community neighbours of a node $u \in C_i$ to be any node $v \notin C_i$ that has an edge pointing to u.

Theorem 3. *Let the observed graph be $\widetilde{A} = A + E$ with K communities and the perturbation E denotes the edges amongst communities $C_1, \cdots, C_K$. Assume that the norm of E satisfies the conditions in Lemma 5. For node $u \in C_i$, let Γ_u^j denote its set of neighbors in C_j for $j \in (1, \cdots, K)$. The spectral coordinate $\boldsymbol{\alpha}_u$ can be approximated as:*

$$\boldsymbol{\alpha}_u \approx x_{iu} I_i + \left(\sum_{j=1}^n \nabla_{uj} \sum_{v \in \Gamma_v^1} \frac{e_{jv} x_{1v}}{\lambda_1}, \cdots, \sum_{j=1}^n \nabla_{uj} \sum_{v \in \Gamma_v^K} \frac{e_{jv} x_{Kv}}{\lambda_K}\right), \tag{11}$$

where I_i is the i-th row of a K-by-K identity matrix, e_{jv} is the (j, v) entry of E and ∇ is defined in Theorem 1.

The entry $\sum_{j=1}^n \nabla_{uj} \sum_{v \in \Gamma_v^i} \frac{e_{jv} x_{iv}}{\lambda_i}$ in the i-th column position of the spectral coordinate in Equation (11) is responsible for determining the influence of the perturbation to the current community members. For general perturbation, the perturbation could occur inside the community or even onto the node itself. Therefore, for the spectral coordinates of node u, this term will be 0 only when the perturbation does not appear on the column positions of E corresponding to the community which the node u belongs to. On the other hand, if perturbations occur inside the column positions of E corresponding to the community where the node u is, the values of $\boldsymbol{\alpha}_{iu} (\forall u \in C_i)$ will be altered. This phenomenon is reasonable, since all members in a community are strongly connected. Hence, the perturbation influence flows from one community member to another.

With Theorem 3, we can make the following analysis and explain some of the phenomena observed from Figure 2 and Table I. Before the perturbation, when the adjacency matrix A is of the diagonal block form, the second part of right hand side of Equation (11) will be 0, so nodes from the community C_i will lie on line I_i. Since $I_i \cdot I_m = 0$ for $i \neq m$, the nodes from different communities lie on different orthogonal lines. After the matrix is perturbed, suppose that the perturbation happens on the C_m region of the vth column of E, then $E\boldsymbol{x}_i = 0$, since the C_m region of $\boldsymbol{x}_i = 0$ by (9). Then the coordinates of all the nodes in C_i with respect to the two-dimensional subspace become $(\boldsymbol{x}_{iu}, \sum_{j=1}^n \nabla_{uj} \sum_{v \in \Gamma_v^m} \frac{e_{jv} x_{mv}}{\lambda_m})$ for $j \in (1, \cdots, n)$ and $u \in C_i$. Likewise, the coordinates of the nodes in C_m are: $(0, \boldsymbol{x}_{mw} + \sum_{j=1}^n \nabla_{wj} \sum_{v \in \Gamma_v^m} \frac{e_{jv} x_{mv}}{\lambda_m})$ for $j \in (1, \cdots, n)$ and $w \in C_m$. The dot product of any two rows is not 0, so the projections of nodes do not form strict orthogonal lines. Due to the sum of the product of scalar ∇_{ij} and the remaining terms, the spectral projections of all the nodes u of the same community C_i onto the sub-eigenspace will deviate from the original line at different rates depending on the values in ∇.

The following is an example with a graph of two communities to illustrate the above proposition. Suppose nodes u and v are from community C_1 and C_2 respectively, the perturbation matrix E add an edge from u to v as $u \rightarrow v$. Then the spectral coordinates for nodes u and v in the two dimensional space would be:

$$\begin{pmatrix} \boldsymbol{x}_{1u} & \nabla_{uu} \frac{e_{uv}}{\lambda_2} x_{2v} \\ \mathbf{0} & \boldsymbol{x}_{2v} + \nabla_{vu} \frac{e_{uv}}{\lambda_2} x_{2v} \end{pmatrix}. \tag{12}$$

This result coincides precisely with the observed pattern in the case in which and edge is added from node 10 to node 5 in Figure 2 and the data shown in Table I.

D. Perturbation Influences to Eigenvectors

Although the components in the perturbed eigenvectors would not change much under ideal conditions, but the situation will be extremely complicated in a very large network with communities of various sizes and with a relatively large amount of intercommunity edges E. The authors [6] states that, for undirected graphs, under the condition that $\|U^T E U\|_2 \leq \|E\|_2$ and $\lambda_i - \lambda_{i+1}$ is greater than $3\|E\|_2$, the projections form quasi orthogonal lines where nodes lie in the vicinity of a fitted line. In general, for both directed and undirected graphs, when $\|E\|_2$ is small, nodes from the same community will form a cluster towards a fitted line by using the spectral projection method according to our theories above. Therefore, nodes can be clustered in the spectral space based on their spectral coordinates when the norm of E is small, but the clustering results would degrade as E grows.

Another factor that could completely alter the eigenvectors is that when a larger community points an edge to a smaller community, the Perron-Frobenius eigenvector corresponding to the smaller community will be altered tremendously, since the inflow perturbation from the eigenvector corresponding to the larger spectral radius completely dominates the perturbed eigenvectors of smaller community.

The following corollary will be used to explain mathematically the reasons for observed sign changes in eigenvectors from section IV B:

Corollary 1. *The sign changes in perturbed eigenvectors $\tilde{\boldsymbol{x}}_i$ are influenced by the corresponding spectral radius λ_j of the neighbouring community C_j. There are three cases:*

1) *The values in $\tilde{\boldsymbol{x}}_i$ remains same signed when $\lambda_i \gg \lambda_j$.*
2) *The values in $\tilde{\boldsymbol{x}}_i$ will be mixed signed when $\lambda_i \leq \lambda_j$.*

Another situation needs to be considered is the fourth case in Figure 2, where two communities were joint by an undirected edge to become a single strongly connected component. In this case, the perturbed eigenvectors corresponding to the larger spectral radius will become the Perrons-Frobenius eigenvector of the newly formed strongly connected component, thus if will be strictly positive. However, the eigenvector corresponding to the smaller community will be mixed signed by Lemma 2.

As indicated by the observations in section IV B, these results can be used to explain the behaviors of nodes in the spectral space mathematically. At this moment, what will happen in the middle between the two intervals in the above theory depends on various conditions including $\|E\|$, $\|C_i\|$ and $\|C_j\|$, thus remains unclear. However, further analysis of such situations is not the focus of this paper and it shall be included in future works.

E. The Core of a Community

At the beginning of this paper, we started our study from communities that have strongly connected graphs. However, in application, this requirement is overly strict. Hence, we seek a way to relieve this restriction. Based on the observations from Figure 2 and Table I, we noticed that the node labeled as 11 can be clustered into the smallest community without being part of the strongly connected component corresponding to that community. This phenomenon can only happen when there exist at least an edge pointing towards the community from the node. This can be justified using Proposition 12 and Equation (12), where the positions without incoming perturbation will remain the same values as before.

When it comes to partition a directed graph, it is necessary to distinguish between the meanings of incoming and outgoing edges. In general, we should consider the node belongs to some community if it has edges pointing to that community. In the cases where one node has edges pointing to multiple communities, the node will be clustered to the one having the most edge weights according to the results in Theorem 3 and example 12. As for the communities, our assumption is that if a community is dense, then the probability is high for it to have a small portion of leaf nodes and one very large strongly connected component which we informally denote as the core of a community. Hence, based on the matrix perturbation theory, all the other nodes that has edges pointing to such communities can be clustered into those communities in the spectral subspace which we studied in this work. As a result, when we combined these observations with Theorem 3 and the example demonstrated in Equation 12, we have the following conclusion:

Theorem 4. *Suppose that a community has a large core and a small portion of leaf nodes which all has edges pointing to the members in the core. Let the leaf edges be a perturbation matrix E and treat the core as A. If the norm of E satisfies the conditions in Lemma 5, then after the perturbation, the leaf nodes will have values in the corresponding locations of the perturbed Perron-Frobenius eigenvector of A.*

Hence, we can use this theorem to replace the assumption that all the graphs of communities are strongly connected with a weaker one, and propose the following result:

Corollary 2. *Assume that each community in the network has a large core and the network contains a relatively small amount of leaf nodes. Further assume that all the leaf nodes have edges pointing to some members in some core(s). Let the leaf edges be a perturbation matrix E and treat the cores as A. Further assume that the norm of E satisfies the conditions in Lemma 5. Then all the nodes can be clustered according to the their corresponding eigenvalue positions in the perturbed Perron-Frobenius spectral subspace.*

With all the theories complete, one concern remained is that wether the perturbed Perron-Frobenius spectral subspace is still real valued. The issue can be addressed by direct linear algebra results: due to the way how eigenvectors are calculated, the perturbed eigenvectors can be solved for real values if the corresponding eigenvalues were real. In our case, since a reducible matrix contains all the eigenvalues of its irreducible components, the spectral radii are all real valued. As a result, there exist real valued solutions for the corresponding eigenvectors.

V. Perturbed Perron-Frobenius Simple Invariant Subspace

According to Theorem 3, we have $\boldsymbol{q}_i = \boldsymbol{x}_i (i = 1, \cdots, K)$, so $\boldsymbol{x}_i (i = 1, \cdots, K)$ are part of the orthornormal basis. Then by Lemma 2 and Definition 1, $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K)$ is a simple invariant subspace. Hence, all of our results based on the graph perturbation theory can be generalized into the simple invariant subspace spanned by the Perron-Frobenius eigenvectors. When combined with the results of Perron-Frobenius Theorem, this particular simple invariant subspace has many unique properties and contains the most topological information of an adjacency matrix. To be specific, for a directed graph, the perturbed Perron-Frobenius simple invariant subspace has the following properties: it is real valued, values in each column vector are same signed when no incoming perturbation exists, and its dimension equals the number of the communities. Furthermore, the same properties hold true for undirected graphs, since those are just special cases for directed graphs.

The reason that each individual column vector without incoming perturbation or with small perturbation in this subspace is strictly same signed rather than all positive is that although Lemma 2 states that each Perron-Frobenius eigenvector is the only all positive vector for each individual community, but the signs of the eigenvectors corresponding to the spectral radii can change after the perturbation. In chapter 2 section 3, the book [19] stated the conditions under which the sign of Perron-Frobenius eigenvector can flip when strongly connected components are linked by edges. To summarize, in a reducible matrix, when a smaller irreducible component has at least an edge pointing to larger irreducible component, the sign of the Perron-Frobenius eigenvector corresponding to the smaller component will change to the opposite of its original.

Therefore, it seems promising to generalize some properties as follows: when two communities are connected by edges that do not alter the macrostructure of the network and when E is small, the signs of the components in the perturbed Perron-Frobenius eigenvectors will remain the same. However, when two communities are connected by edge that merge them into one single strongly connected component, the signs of the values in the Perron-Frobenius eigenvectors corresponding to the smaller community will become mixed while the Perron-Frobenius eigenvector corresponding to the bigger community is strictly non-negative or non-positive. As a matter of fact, the perturbed eigenvector corresponding the the larger spectral radius is the new Perron-Frobenius eigenvector representing the newly formed community. In summery, the perturbed Perron-Frobenius simple invariance subspace is capable of correctly capturing the features of communities after changes to the structures have occurred. Therefore, we can leverage those properties of such a subspace to analyze the structure of the network in the spectral domain.

At this point, based on the theories developed, we explored the adjacency eigenspace properties of directed graphs directly and proposed two criteria to look for the eigenvectors that can be used to perform spectral projections. These findings and theoretical results provided us some guidelines on selecting candidate spectral subspace for clustering in directed graphs.

VI. Conclusion and Future Work

In this research, the properties of the adjacency eigenspaces of directed graphs were investigated. We started our theoretical work from networks with disconnected communities and made a strong assumption that each community should have a strongly connected graph. By using the matrix perturbation theory, we constructed an orthonormal basis that contained the Perron-Frobenius eigenvectors corresponding to all the communities to achieve the orthonormal reduction of the adjacency matrices of directed graphs and described mathematically how the projections of nodes would behave in the perturbed Perron-Frobenius simple invariant subspace. All the mathematical results and our theories coincide with the observations in the example provided in section IV B. Then, based on our theories and mathematical results that explained why leaf nodes can be clustered in to the core of a community, we replaced the original assumption that all the communizes should be strongly connected with a much weaker one that only requires a community to have a strongly connected core component. As a result, the theoretical results could explain various phenomena in the adjacency eigenspace precisely. However, due to limited space, this work will only emphasis on the theories and the applications would be included in future works. The theories provided in this work can be extended to analyze various graph related problems including but not limited to: studying the microstructure of a community, analyzing the changes of the macrostructure of a network, fraud detections and extending the applications of current theories to non-square matrices.

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APPENDIX A
PROOFS

A. Proof of Proposition 1

Proof: For 1), since (X, Q) is the orthonormal basis formed by Gram-Schmidt process, so $(X, Q)^H = (\boldsymbol{x}_i, Q)^{-1}$ and it is unitary.

For 2), Since X are eigenvectors of A, then $Q^H A\boldsymbol{x}_i = Q^H \lambda_i \boldsymbol{x}_i = \lambda_i \mathbf{0} = \mathbf{0}$ for each $i \in (1, \cdots, K)$. Since λ_is are simple eigenvalues, then $\mathcal{L}(L_1) \cap \mathcal{L}(L_2) = \emptyset$. By Lemma 3 and Definition 1, $\Re(X)$ is a simple invariant subspace of A. By Lemma 4, A can be reduced to a block triangular form in (4). Hence, Equation (6) holds. By the method how (X, Q) was formed, due to the mechanism of Gram-Schmidt process, we have $(X, Q)R = \mathbf{x}$ where R and R^{-1} is strick upper triangular. Hence, $(X, Q)^H A(X, Q) = R\Lambda R^{-1}$ is the result of the orthonormal reduction and is upper triangular, then so is L_2. The last part is the direct result from Lemma 4. ■

B. Proof of Theorem 1

Proof: If the eigenvectors are part of the unitary orthonormal basis and they form a simple invariant subspace of A as the conditions in Proposition 1, by applying Lemma 5 directly, we have the above result. ■

C. Proof of Theorem 2

Proof: Noticing that the eigenvectors corresponding to the Perron-Frobenius eigenvalues are orthogonal before the perturbation occurs, so if we construct an orthonormal basis as in Proposition 1, those eigenvectors are part of the orthonormal basis and $\boldsymbol{q}_i = \boldsymbol{x}_i (i = 1, \cdots, K)$, where the indexes are relabeled to correspond to each community.

Let the Perron-Frobinues eigenvectors be put together as $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K)$. By Lemma 2, all the eigenvalues corresponding to such a spectral subspace are simple. Hence, by Definition 1 and Proposition 1, this spectral subspace is a simple invariant subspace of A.

By Theorem 1, each of the perturbed K eigenvectors corresponding to the Perron-Frobenius eigenvalues can be approximated as:

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + \nabla E \frac{\boldsymbol{x}_i}{\lambda_i}.$$

When the K columns are put together to form the perturbed spectral subspace, the approximation of such a perturbed spectral space has the form:

$$(\tilde{\boldsymbol{x}}_1, \cdots, \tilde{\boldsymbol{x}}_K) \approx (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K) + \nabla E(\frac{\boldsymbol{x}_1}{\lambda_1}, \cdots, \frac{\boldsymbol{x}_K}{\lambda_K}).$$

■

D. Proof of Theorem 3

Proof: By Theorem 2, the perturbed spectral space have the form:

$$(\tilde{\boldsymbol{x}}_1, \cdots, \tilde{\boldsymbol{x}}_K) \approx (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_K) + \nabla E(\frac{\boldsymbol{x}_1}{\lambda_1}, \cdots, \frac{\boldsymbol{x}_K}{\lambda_K}).$$

Then, by Lemma 6,and Equation (9), the spectral coordinate of node u can be simplified by only considering the influences by neighbours from other communities:

$$\begin{aligned}\boldsymbol{\alpha}_u &\approx x_{iu}(0, \cdots, 1_i, \cdots, 0) \\ &+ \left(\sum_{j=1}^{n} \nabla_{uj} \sum_{v \in C_1} \frac{e_{jv} x_{1v}}{\lambda_1}, \cdots, \sum_{j=1}^{n} \nabla_{uj} \sum_{v \in C_K} \frac{e_{jv} x_{Kv}}{\lambda_K} \right) \\ &\approx x_{iu} I_i \\ &+ \left(\sum_{j=1}^{n} \nabla_{uj} \sum_{v \in \Gamma_v^1} \frac{e_{jv} x_{1v}}{\lambda_1}, \cdots, \sum_{j=1}^{n} \nabla_{uj} \sum_{v \in \Gamma_v^K} \frac{e_{jv} x_{Kv}}{\lambda_K} \right),\end{aligned}$$

where I_i is the i-th row of a K-by-K identity matrix. ■

E. Proof of Corollary 1

Proof: A simplified mathematical representation for this phenomena can be explained by using Equation (7).The perturbed eigenvector of it can be approximated as:

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + Q(I - \frac{L_2}{\lambda_i})^{-1} Q^H E \frac{\boldsymbol{x}_i}{\lambda_i}.$$

Since the diagonal elements of L_2 corresponds to spectral radii of neighbouring communities according to Proposition 1, then the term $(I - \frac{L_2}{\lambda_i})^{-1}$ would be determined by the incoming influences from other communities.

In the first case, when $\lambda_i \gg \lambda_j$, the perturbation will have minimal influence, thus the values in $\tilde{\boldsymbol{x}}_i$ will not be perturbed too much and will keep their original signs.

In the later case, the perturbation results will be completely dominated by the incoming influence, so $(I - \frac{L_2}{\lambda_i})$ will contain all negative values on its diagonal. Therefore, most of the locations in $\tilde{\boldsymbol{x}}_i$ corresponding to those of nodes in C_j will have different signed values than the original. This caused the values of $\tilde{\boldsymbol{x}}_i$ to be mixed signed. ■

F. Proof of Theorem 4

Proof: By applying Theorem 1 and Theorem 3, the values are $\sum_{j=1}^{n} \nabla_{vj} \frac{e_{vj} x_{iv}}{\lambda_i}$). ■

G. Proof of Corollary 2

Proof: Since the norm of E satisfies the conditions in Lemma 5. By Theorem 4, all the nodes will have values in the corresponding positions of the perturbed Perron-Frobenius eigenvectors. Therefore, nodes can be clustered according to the their corresponding eigenvectors in the perturbed Perron-Frobenius spectral subspace. ■